Solutions to *Modern Quantum Mechanics* (J. J. Sakurai)

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Problem 1.1

Prove $[AB, CD] = -AC\{D, B\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB$.

Start with $-AC\{D, B\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB$.

$= -AC(DB + BD) + A(CB + BC)D - C(DA + AD)B + (CA + AC)DB$

$= -ACDB - ACBD + ACBD + ABCD - CDAB - CADB + CADB + ACDB$

$= ABCD - CDAB$

$= [AB, CD]$

$\implies \boxed{[AB, CD] = -AC\{D, B\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB}$ ■

Problem 1.2

Suppose a 2×2 matrix X (not necessarily Hermitian, nor unitary) is written as $X = a_0 + \boldsymbol{\sigma} \cdot \boldsymbol{a}$, where a_0 and $a_{1,2,3}$ are numbers.

(a) How are a_0 and a_k $(k = 1, 2, 3)$ related to $\text{tr}(X)$ and $\text{tr}(\sigma_k X)$?

We can write $X = a_0 + \boldsymbol{\sigma} \cdot \boldsymbol{a}$

$= a_0 + \sum_{i=1}^{3} \sigma_i a_i$

$= a_0 + \sigma_1 a_1 + \sigma_2 a_2 + \sigma_3 a_3$

We can also represent σ_i as the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \;\; \sigma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \;\; \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\text{tr}(X) = a_0 \text{tr}(\mathbf{1}) + \sum_{i=1}^{3} \text{tr}(\sigma_i) a_i$

$$\text{tr}(X) \doteq \text{tr} \begin{pmatrix} a_0 & 0 \\ 0 & a_0 \end{pmatrix} + \text{tr} \begin{pmatrix} 0 & a_1 \\ a_1 & 0 \end{pmatrix} + \text{tr} \begin{pmatrix} 0 & -ia_2 \\ ia_2 & 0 \end{pmatrix} + \text{tr} \begin{pmatrix} a_3 & 0 \\ 0 & -a_3 \end{pmatrix}$$

$$\text{tr}(X) = \text{tr} \begin{pmatrix} a_0 & 0 \\ 0 & a_0 \end{pmatrix}$$

$\boxed{\text{tr}(X) = 2a_0}$

$\text{tr}(\sigma_k X) = a_0 \text{tr}(\sigma_k) + \sum_{i=1}^{3} \text{tr}(\sigma_i \sigma_k) a_i$

$\mathrm{tr}(\sigma_k X) = a_0 \mathrm{tr}(\sigma_k) + \frac{1}{2}\sum_{i=1}^{3} \mathrm{tr}(\sigma_i \sigma_k + \sigma_k \sigma_i) a_i$

$\mathrm{tr}(\sigma_k X) = a_0 \mathrm{tr}(\sigma_k) + \sum_{i=1}^{3} \mathrm{tr}(\delta_{ik}) a_i$

$\mathrm{tr}(\sigma_k X) = a_0 \mathrm{tr}(\sigma_k) + \sum_{i=1}^{3} \delta_{ik} \mathrm{tr}(\mathbf{1}) a_i$

$\mathrm{tr}(\sigma_k X) = \mathrm{tr}(\mathbf{1}) a_k$

$\boxed{\mathrm{tr}(\sigma_k X) = 2a_k}$

(b) Obtain a_0 and a_k in terms of the matrix elements X_{ij}.

The observable X can be represented by a 2×2 matrix $X_{ij} \doteq \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$.

We have that the $\mathrm{tr}(X) = 2a_0$

$\implies X_{11} + X_{22} = 2a_0$

$\implies \boxed{a_0 = \frac{1}{2}(X_{11} + X_{22})}$

We also have that the $\mathrm{tr}(\sigma_k X) = 2a_k$, so for each $k = 1, 2, 3$ we can write

$$\mathrm{tr}(\sigma_1 X) = \mathrm{tr} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

$$= \mathrm{tr} \begin{pmatrix} X_{21} & X_{22} \\ X_{11} & X_{12} \end{pmatrix}$$

$= X_{21} + X_{12} = 2a_1$

$\implies \boxed{a_1 = \frac{1}{2}(X_{21} + X_{12})}$

$$\mathrm{tr}(\sigma_2 X) = \mathrm{tr} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

$$= \mathrm{tr} \begin{pmatrix} -iX_{21} & -iX_{22} \\ iX_{11} & iX_{12} \end{pmatrix}$$

$= -iX_{21} + iX_{12} = 2a_2$

$\implies \boxed{a_2 = -\frac{i}{2}(X_{21} + X_{12})}$

$$\mathrm{tr}(\sigma_3 X) = \mathrm{tr} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} \\ -X_{21} & -X_{22} \end{pmatrix}$$

$$= \text{tr}\begin{pmatrix} -iX_{21} & -iX_{22} \\ iX_{11} & iX_{12} \end{pmatrix}$$

$$= X_{11} - X_{22} = 2a_3$$

$$\implies \boxed{a_3 = \frac{1}{2}(X_{11} - X_{22})} \quad \blacksquare$$

Problem 1.3

Problem 1.4

Using the rules of bra-ket algebra, prove or evaluate the following:

(a) $\text{tr}(XY) = \text{tr}(YX)$, where X and Y are operators;

$\text{tr}(XY) = \sum_{a'} \langle a'|XY|a'\rangle$

$\text{tr}(XY) = \sum_{a'} \sum_{a''} \langle a'|X|a''\rangle\langle a''|Y|a'\rangle$

$\text{tr}(XY) = \sum_{a'} \sum_{a''} \langle a''|Y|a'\rangle\langle a'|X|a''\rangle$

$\text{tr}(XY) = \sum_{a''} \langle a''|YX|a''\rangle$

$\boxed{\text{tr}(XY) = \text{tr}(YX)}$

(b) $(XY)^\dagger = Y^\dagger X^\dagger$, where X and Y are operators;

$Y|\alpha\rangle \overset{DC}{\leftrightarrow} \langle\alpha|Y^\dagger$

$X(Y|\alpha\rangle) = XY|\alpha\rangle \overset{DC}{\leftrightarrow} \langle\alpha|Y^\dagger X^\dagger = (\langle\alpha|Y^\dagger)X^\dagger$

$\implies \boxed{XY = Y^\dagger X^\dagger}$

(c) $\exp[if(A)] = ?$, where A is a Hermitian operator whose eigenvalues are known;

$\exp(\mathbf{A}) = \mathbf{1} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \frac{1}{3!}\mathbf{A}^3 + \cdots$

$\exp[if(\mathbf{A})] = \mathbf{1} + if(\mathbf{A}) + \frac{1}{2!}[if(\mathbf{A})]^2 + \cdots$

$\boxed{\exp[if(\mathbf{A})] \simeq \mathbf{1} + if(\mathbf{A})}$ (to first order)

(d) $\sum_{a'} \psi^*_{a'}(\mathbf{x}')\psi_{a'}(\mathbf{x}'')$, where $\psi_{a'}(\mathbf{x}') = \langle\mathbf{x}'|a'\rangle$.

$\sum_{a'} \psi^*_{a'}(\mathbf{x}')\psi_{a'}(\mathbf{x}'') = \sum_{a'} \langle a'|\mathbf{x}'\rangle\langle\mathbf{x}''|a'\rangle$

$\sum_{a'} \psi^*_{a'}(\mathbf{x}')\psi_{a'}(\mathbf{x}'') = \sum_{a'} \langle \mathbf{x}''|a'\rangle\langle a'|\mathbf{x}'\rangle$

$\sum_{a'} \psi^*_{a'}(\mathbf{x}')\psi_{a'}(\mathbf{x}'') = \langle \mathbf{x}''|\mathbf{x}'\rangle$

$$\boxed{\sum_{a'} \psi^*_{a'}(\mathbf{x}')\psi_{a'}(\mathbf{x}'') = \delta(\mathbf{x}'' - \mathbf{x}')} \quad \blacksquare$$

Problem 1.5

(a) Consider two kets $|\alpha\rangle$ and $|\beta\rangle$. Suppose $\langle a'|\alpha\rangle$, $\langle a''|\alpha\rangle$, $\ldots$ and $\langle a'|\beta\rangle$, $\langle a''|\beta\rangle$, $\ldots$ are all known, where $|a'\rangle$, $|a''\rangle$, $\ldots$ form a complete set of base kets. Find the matrix representation of the operator $|\alpha\rangle\langle\beta|$ in that basis.

We have a complete set of base kets $\{|a^{(n)}\rangle\} \longrightarrow \{|a'\rangle, |a''\rangle, |a'''\rangle, \ldots\}$. $|\alpha\rangle, |\beta\rangle \longrightarrow \langle a'|\alpha\rangle, \langle a''|\alpha\rangle, \ldots$ and $\langle a'|\beta\rangle, \langle a''|\beta\rangle, \ldots$ are known.

We define an operator $X \equiv |\alpha\rangle\langle\beta|$.

$$\langle a^{(n)}|X|a^{(m)}\rangle = \langle a^{(n)}|\alpha\rangle\langle\beta|a^{(m)}\rangle \doteq \boxed{\begin{pmatrix} \langle a'|\alpha\rangle\langle\beta|a'\rangle & \langle a'|\alpha\rangle\langle\beta|a''\rangle & \cdots \\ \langle a''|\alpha\rangle\langle\beta|a'\rangle & \langle a''|\alpha\rangle\langle\beta|a''\rangle & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}}$$

(b) We now consider a spin $\frac{1}{2}$ system and let $|\alpha\rangle$ and $|\beta\rangle$ be $|s_z = \hbar/2\rangle$ and $|s_x = \hbar/2\rangle$, respectively. Write down explicitly the square matrix that corresponds to $|\alpha\rangle\langle\beta|$ in the usual (s_z diagonal) basis.

$|\alpha\rangle \longrightarrow |s_z = \frac{\hbar}{2}\rangle = |+\rangle$

$|\beta\rangle \longrightarrow |s_x = \frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle \implies \langle\beta| \longrightarrow \langle s_x = \frac{\hbar}{2}| = \frac{1}{\sqrt{2}}\langle+| + \frac{1}{\sqrt{2}}\langle-|$

$\langle a^{(n)}|X|a^{(m)}\rangle = \langle a^{(n)}|\alpha\rangle\langle\beta|a^{(m)}\rangle \longrightarrow \langle a^{(n)}|s_z = \frac{\hbar}{2}\rangle\langle s_x = \frac{\hbar}{2}|a^{(m)}\rangle$

$$\doteq \begin{pmatrix} \langle+|s_z;+\rangle\langle s_x;+|+\rangle & \langle+|s_z;+\rangle\langle s_x;+|-\rangle \\ \langle-|s_z;+\rangle\langle s_x;+|+\rangle & \langle-|s_z;+\rangle\langle s_x;+|-\rangle \end{pmatrix} = \begin{pmatrix} \langle+|+\rangle\langle s_x;+|+\rangle & \langle+|+\rangle\langle s_x;+|-\rangle \\ \langle-|+\rangle\langle s_x;+|+\rangle & \langle-|+\rangle\langle s_x;+|-\rangle \end{pmatrix} = \boxed{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}} \quad \blacksquare$$

Problem 1.6

Suppose $|i\rangle$ and $|j\rangle$ are eigenkets of some Hermitian operator A. Under what condition can we conclude that $|i\rangle + |j\rangle$ is also an eigenket of A? Justify your answer.

$A|i\rangle = i|i\rangle$

and

$A|j\rangle = j|j\rangle$

$\implies A(|i\rangle + |j\rangle)$

$= A|i\rangle + A|j\rangle$

$= i|i\rangle + j|j\rangle$

$|k\rangle \equiv |i\rangle + |j\rangle$

$\implies \boxed{A|k\rangle = k|k\rangle \neq i|i\rangle + j|j\rangle \text{ unless the eigenvalues are degenerate (i.e. } i = j)}$ ■

Problem 1.8

Using the orthonormality of $|+\rangle$ and $|-\rangle$, prove $[S_i, S_j] = i\epsilon_{ijk}\hbar S_k$ and $\{S_i, S_j\} = \left(\frac{\hbar^2}{2}\right)\delta_{ij}$ where

$S_x = \frac{\hbar}{2}(|+\rangle\langle-| + |-\rangle\langle+|)$

$S_y = \frac{i\hbar}{2}(-|+\rangle\langle-| + |-\rangle\langle+|)$

$S_z = \frac{\hbar}{2}(|+\rangle\langle+| - |-\rangle\langle-|)$

$[S_x, S_x] = S_xS_x - S_xS_x$

$= \frac{\hbar^2}{4}\Big(|+\rangle\langle-| + |-\rangle\langle+|\Big)\Big(|+\rangle\langle-| + |-\rangle\langle+|\Big) - \frac{\hbar}{4}\Big(|+\rangle\langle-| + |-\rangle\langle+|\Big)\Big(|+\rangle\langle-| + |-\rangle\langle+|\Big)$

$= \frac{\hbar^2}{4}\Big(|+\rangle\langle-|+\rangle\langle-| + |-\rangle\langle+|+\rangle\langle-| + |+\rangle\langle-|-\rangle\langle+| + |-\rangle\langle+|-\rangle\langle+|\Big)$

$-\frac{\hbar^2}{4}\Big(|+\rangle\langle-|+\rangle\langle-| + |-\rangle\langle+|+\rangle\langle-| + |+\rangle\langle-|-\rangle\langle+| + |-\rangle\langle+|-\rangle\langle+|\Big)$

$= \frac{\hbar^2}{4}\Big(|-\rangle\langle-| + |+\rangle\langle+|\Big) - \frac{\hbar^2}{4}\Big(|-\rangle\langle-| + |+\rangle\langle+|\Big)$

$= \boxed{0}$

$[S_x, S_y] = S_xS_y - S_yS_x$

$= \frac{i\hbar^2}{4}\Big(|+\rangle\langle-| + |-\rangle\langle+|\Big)\Big(-|+\rangle\langle-| + |-\rangle\langle+|\Big) - \frac{i\hbar^2}{4}\Big(-|+\rangle\langle-| + |-\rangle\langle+|\Big)\Big(|+\rangle\langle-| + |-\rangle\langle+|\Big)$

$= \frac{i\hbar^2}{4}\Big(-|+\rangle\langle-|+\rangle\langle-| - |-\rangle\langle+|+\rangle\langle-| + |+\rangle\langle-|-\rangle\langle+| + |-\rangle\langle+|-\rangle\langle+|\Big)$

$-\frac{i\hbar^2}{4}\Big(-|+\rangle\langle-|+\rangle\langle-| + |-\rangle\langle+|+\rangle\langle-| - |+\rangle\langle-|-\rangle\langle+| + |-\rangle\langle+|-\rangle\langle+|\Big)$

$= \frac{i\hbar^2}{4}\Big(|+\rangle\langle+| - |-\rangle\langle-|\Big) - \frac{i\hbar^2}{4}\Big(-|+\rangle\langle+| + |-\rangle\langle-|\Big)$

$= \frac{i\hbar^2}{2}\Big(|+\rangle\langle+| - |-\rangle\langle-|\Big)$

$= \boxed{i\hbar S_z}$

$[S_x, S_z] = S_xS_z - S_zS_x$

$= \frac{\hbar^2}{4}\Big(|+\rangle\langle-| + |-\rangle\langle+|\Big)\Big(|+\rangle\langle+| - |-\rangle\langle-|\Big) - \frac{\hbar^2}{4}\Big(|+\rangle\langle+| - |-\rangle\langle-|\Big)\Big(|+\rangle\langle-| + |-\rangle\langle+|\Big)$

$= \frac{\hbar^2}{4}\Big(|+\rangle\langle-|+\rangle\langle+| + |-\rangle\langle+|+\rangle\langle+| - |+\rangle\langle-|-\rangle\langle-| - |-\rangle\langle+|-\rangle\langle-|\Big)$

$-\frac{\hbar^2}{4}\Big(|+\rangle\langle+|+\rangle\langle-| - |-\rangle\langle-|+\rangle\langle-| + |+\rangle\langle+|-\rangle\langle+| - |-\rangle\langle-|-\rangle\langle+|\Big)$

$= \frac{\hbar^2}{4}\Big(|-\rangle\langle+| - |+\rangle\langle-|\Big) - \frac{\hbar^2}{4}\Big(|+\rangle\langle-| - |-\rangle\langle+|\Big)$

$= \frac{\hbar^2}{2}\Big(|-\rangle\langle+| - |+\rangle\langle-|\Big)$

$= \frac{-i\hbar}{2} \cdot i\hbar\Big(-|+\rangle\langle-| + |-\rangle\langle+|\Big)$

$= \boxed{-i\hbar S_y}$

Problem 1.10

The Hamiltonian operator for a two-state system is given by $H = a(|1\rangle\langle1| - |2\rangle\langle2| + |1\rangle\langle2| + |2\rangle\langle1|)$, where a is a number with the dimension of energy. Find the energy eigenvalues and the corresponding energy eigenkets (as linear combinations of $|1\rangle$ and $|2\rangle$).

$H = a(|1\rangle\langle1| - |2\rangle\langle2| + |1\rangle\langle2| + |2\rangle\langle1|)$

$H|\alpha\rangle = \lambda|\alpha\rangle$ with $|\alpha\rangle = A|1\rangle + B|2\rangle$

$H|\alpha\rangle = a(|1\rangle\langle1| - |2\rangle\langle2| + |1\rangle\langle2| + |2\rangle\langle1|)(A|1\rangle + B|2\rangle)$

Problem 1.18

(a) The simplest way to derive the Schwarz inequality goes as follows. First, observe

$(\langle\alpha| + \lambda^*\langle\beta|) \cdot (|\alpha\rangle + \lambda|\beta\rangle) \geq 0$

for any complex number λ; then choose λ in such a way that the preceding inequality reduces to the Schwarz inequality.

$(\langle\alpha| + \lambda^*\langle\beta|) \cdot (|\alpha\rangle + \lambda|\beta\rangle) \geq 0$

$\lambda = -\frac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}$

$\langle\alpha|\alpha\rangle + \lambda^*\langle\beta|\alpha\rangle + \lambda\langle\alpha|\beta\rangle + |\lambda|^2\langle\beta|\beta\rangle \geq 0$

$\langle\alpha|\alpha\rangle - \frac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}\langle\beta|\alpha\rangle - \frac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}\langle\alpha|\beta\rangle + \frac{|\langle\beta|\alpha\rangle|^2}{\langle\beta|\beta\rangle^2}\langle\beta|\beta\rangle \geq 0$

$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle - |\langle\alpha|\beta\rangle|^2 - |\langle\alpha|\beta\rangle|^2 + |\langle\alpha|\beta\rangle|^2 \geq 0$

$\implies \boxed{\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2}$

(b) Show that the equality sign in the generalized uncertainty relation holds if the state in question satisfies

$\Delta A|\alpha\rangle = \lambda \Delta B|\alpha\rangle$

with λ purely *imaginary*.

The generalized uncertainty relation is

$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq |\langle[A,B]\rangle|^2$

$\Delta A|\alpha\rangle = \lambda \Delta B|\alpha\rangle \longleftrightarrow \lambda^*\langle\alpha|\Delta B + \langle\alpha|\Delta A$

$\langle(\Delta A)^2\rangle = \langle\Delta A \cdot \Delta A\rangle = \langle\alpha|\Delta A \cdot \Delta A|\alpha\rangle$

$\langle(\Delta A)^2\rangle = \lambda^*\langle\alpha|\Delta B \cdot \lambda\Delta B|\alpha\rangle$

$\langle(\Delta A)^2\rangle = |\lambda|^2\langle\alpha|(\Delta B)^2|\alpha\rangle$

$\langle(\Delta A)^2\rangle = |\lambda|^2\langle(\Delta B)^2\rangle$

$\implies \langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}|\langle[A,B]\rangle|^2$

$|\lambda|^2\langle(\Delta B)^2\rangle \geq \frac{1}{4}|\lambda^* - \lambda|^2\langle(\Delta B)^2\rangle^2$

$|\lambda|^2 \geq \frac{1}{4}|\lambda^* - \lambda|^2$

λ imaginary $\longrightarrow \lambda = -\lambda^*$

Check: $|\lambda|^2 \geq \frac{1}{4}|-2\lambda|^2$

$\boxed{|\lambda|^2 \geq |\lambda|^2}$

(c) Explicit calculations using the usual rules of wave mechanics show that the wave function for a Gaussian wave packet given by

$\langle x'|\alpha\rangle = (2\pi d^2)^{-1/4} \exp\left[\frac{i\langle p\rangle x'}{\hbar} - \frac{(x'-\langle x\rangle)^2}{4d^2}\right]$

satisfies the minimum uncertainty relation

$\sqrt{\langle(\Delta x)^2\rangle}\sqrt{\langle(\Delta p)^2\rangle} = \frac{\hbar}{2}$

Prove that the requirement

$\langle x'|\Delta x|\alpha\rangle = \text{(imaginary number) } \langle x'|\Delta p|\alpha\rangle$

is indeed satisfied for such a Gaussian wave packet, in agreement with (b).

$\langle x'|\Delta x|\alpha\rangle = \langle x'|x - \langle x\rangle|\alpha\rangle$

$\langle x'|\Delta x|\alpha\rangle = \langle x'|x|\alpha\rangle - \langle x\rangle\langle x'|\alpha\rangle$

$\langle x'|\Delta p|\alpha\rangle = \langle x'|p - \langle p\rangle|\alpha\rangle$

$\langle x'|\Delta p|\alpha\rangle = \langle x'|p|\alpha\rangle - \langle p\rangle\langle x'|\alpha\rangle$

$\langle x'|\Delta p|\alpha\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x'}\langle x'|\alpha\rangle - \langle p\rangle\langle x'|\alpha\rangle$

$\langle x'|\Delta p|\alpha\rangle = \frac{\hbar}{i}\left[\frac{i\langle p\rangle}{\hbar} - \frac{(x'-\langle x'\rangle)}{2d^2}\right]\langle x'|\alpha\rangle - \langle p\rangle\langle x'|\alpha\rangle$

$\langle x'|\Delta p|\alpha\rangle = \left[\langle p\rangle - \frac{\hbar}{i}\frac{(x'-\langle x'\rangle}{2d^2} - \langle p\rangle\right]\langle x'|\alpha\rangle$

$\langle x'|\Delta p|\alpha\rangle = \frac{i\hbar}{2d^2}(x' - \langle x'\rangle)\langle x'|\alpha\rangle$

$\langle x'|\Delta p|\alpha\rangle = \frac{i\hbar}{2d^2}\langle x'|(x' - \langle x'\rangle)|\alpha\rangle$

$\langle x'|\Delta p|\alpha\rangle = \frac{i\hbar}{2d^2}\langle x'|\Delta x|\alpha\rangle$

$$\Longrightarrow \boxed{\langle x'|\Delta x|\alpha\rangle = -\frac{2id^2}{\hbar}\langle x'|\Delta p|\alpha\rangle} \quad \blacksquare$$

Problem 1.33

(a) Prove the following:

(i) $\langle p'|x|\alpha\rangle = i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle$

(ii) $\langle\beta|x|\alpha\rangle = \int dp'\phi^*_\beta(p')i\hbar\frac{\partial}{\partial p'}\phi_\alpha(p')$

where $\phi_\alpha(p') = \langle p'|\alpha\rangle$ and $\phi_\beta(p') = \langle p'|\beta\rangle$ and momentum-space wave functions.

Note that $\langle p'|x|p''\rangle = \int\langle p'|x|x'\rangle\langle x'|p''\rangle dx'$

$= \int x'\langle p'|x'\rangle\langle x'|p''\rangle dx'$

$= \frac{1}{2\pi\hbar}\int dx'x' \exp\left[\frac{-ix'(p'-p'')}{\hbar}\right]$

Knowing that $\delta(p'-p'') = \frac{1}{2\pi\hbar}\int dx' \exp\left[\frac{-ix'(p'-p'')}{\hbar}\right]$

$\therefore\ \langle p'|x|p''\rangle = i\hbar\frac{\partial}{\partial p'}\delta(p'-p'')$

$\therefore\ \langle p'|x|\alpha\rangle = \int dp''\langle p'|x|p''\rangle\langle p''|\alpha\rangle$

$= \int dp'' i\hbar\frac{\partial}{\partial p'}\delta(p'-p'')\langle p''|\alpha\rangle$

$$\boxed{= i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle}$$

$\langle\beta|x|\alpha\rangle = \int dp'\langle\beta|p'\rangle\langle p'|x|\alpha\rangle$

$= \int dp'\langle\beta|p'\rangle i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle$

$$= \int dp' \phi_\beta^*(p') i\hbar \frac{\partial}{\partial p'} \phi_\alpha(p')$$

Problem 2.1

Problem 2.13

Consider a one-dimensional simple harmonic oscillator.

(a) Using

$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right)$

$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$

and

$a|n\rangle = \sqrt{n}|n-1\rangle$

$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$

evaluate $\langle m|x|n\rangle$, $\langle m|p|n\rangle$, $\langle m|\{x,p\}|n\rangle$, $\langle m|x^2|n\rangle$, and $\langle m|p^2|n\rangle$.

We can calculate (or look up in the book) the following expressions for x and p,

$x = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(a + a^\dagger)$

$p = i\left(\frac{m\hbar\omega}{2}\right)^{1/2}(a^\dagger - a)$

(i) $\langle m|x|n\rangle$

$x|n\rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(a + a^\dagger)|n\rangle$

$x|n\rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(a|n\rangle + a^\dagger|n\rangle)$

$x|n\rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(\sqrt{n}|n-1\rangle + \sqrt{n+1}|n+1\rangle)$

$\implies \langle m|x|n\rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(\sqrt{n}\langle m|n-1\rangle + \sqrt{n+1}\langle m|n+1\rangle)$

$$\langle m|x|n\rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(\sqrt{n}\delta_{m,n-1} + \sqrt{n+1}\delta_{m,n+1})$$

(ii) $\langle m|p|n\rangle$

$$p|n\rangle = i\left(\frac{m\hbar\omega}{2}\right)^{1/2}(a^\dagger - a)|n\rangle$$

$$p|n\rangle = i\left(\frac{m\hbar\omega}{2}\right)^{1/2}(a^\dagger|n\rangle - a|n\rangle$$

$$p|n\rangle = i\left(\frac{m\hbar\omega}{2}\right)^{1/2}(\sqrt{n+1}|n+1\rangle - \sqrt{n}|n-1\rangle)$$

$$\implies \langle m|p|n\rangle = i\left(\frac{m\hbar\omega}{2}\right)^{1/2}(\sqrt{n+1}\langle m|n+1\rangle - \sqrt{n}\langle m|n-1\rangle)$$

$$\boxed{\langle m|p|n\rangle = i\left(\frac{m\hbar\omega}{2}\right)^{1/2}(\sqrt{n+1}\delta_{m,n+1} - \sqrt{n}\delta_{m,n-1})}$$

(iii) $\langle m|\{x,p\}|n\rangle$

$$\{x,p\} = xp + px$$

$$\{x,p\} = i\left(\frac{\hbar}{2m\omega}\right)^{1/2}\left(\frac{m\hbar\omega}{2}\right)^{1/2}(a+a^\dagger)(a^\dagger - a) + i\left(\frac{\hbar}{2m\omega}\right)^{1/2}\left(\frac{m\hbar\omega}{2}\right)^{1/2}(a^\dagger - a)(a+a^\dagger)$$

$$\{x,p\} = \frac{i\hbar}{2}(-a\cdot a - a^\dagger\cdot a + a\cdot a^\dagger + a^\dagger\cdot a^\dagger) + \frac{i\hbar}{2}(-a\cdot a + a^\dagger\cdot a - a\cdot a^\dagger + a^\dagger\cdot a^\dagger)$$

$$\{x,p\} = i\hbar(a^\dagger\cdot a^\dagger - a\cdot a)$$

$$\{x,p\}|n\rangle = (xp+px)|n\rangle$$

$$\{x,p\}|n\rangle = i\hbar(a^\dagger\cdot a^\dagger - a\cdot a)|n\rangle$$

$$\{x,p\}|n\rangle = i\hbar(a^\dagger\cdot a^\dagger|n\rangle - a\cdot a|n\rangle)$$

$$\{x,p\}|n\rangle = i\hbar(a^\dagger|n+1\rangle\sqrt{n+1} - a|n-1\rangle\sqrt{n})$$

$$\{x,p\}|n\rangle = i\hbar(|n+2\rangle\sqrt{n+1}\sqrt{n+2} - |n-2\rangle\sqrt{n}\sqrt{n-1})$$

$$\implies \langle m|\{x,p\}|n\rangle = i\hbar(\langle m|n+2\rangle\sqrt{n+1}\sqrt{n+2} - \langle m|n-2\rangle\sqrt{n}\sqrt{n-1})$$

$$\boxed{\langle m|\{x,p\}|n\rangle = i\hbar(\sqrt{n+1}\sqrt{n+2}\delta_{m,n+2} - \sqrt{n}\sqrt{n-1}\delta_{m,n-2})}$$

(iv) $x^2 = \frac{\hbar}{2m\omega}(a\cdot a + a^\dagger\cdot a + a\cdot a^\dagger + a^\dagger\cdot a^\dagger)$

$$x^2|n\rangle = \frac{\hbar}{2m\omega}(a\cdot a|n\rangle + a^\dagger\cdot a|n\rangle + a\cdot a^\dagger|n\rangle + a^\dagger\cdot a^\dagger|n\rangle)$$

$$x^2|n\rangle = \frac{\hbar}{2m\omega}(a|n-1\rangle\sqrt{n} + a^\dagger|n-1\rangle\sqrt{n} + a|n+1\rangle\sqrt{n+1} + a^\dagger|n+1\rangle\sqrt{n+1})$$

$$x^2|n\rangle = \frac{\hbar}{2m\omega}(|n-2\rangle\sqrt{n}\sqrt{n-1} + |n\rangle\sqrt{n}\sqrt{n} + |n\rangle\sqrt{n}\sqrt{n+1} + \sqrt{n+2}\sqrt{n+1}|n+2\rangle)$$

$$\implies \langle m|x^2|n\rangle = \frac{\hbar}{2m\omega}(\sqrt{n}\sqrt{n-1}\langle m|n-2\rangle + n\langle m|n\rangle + \sqrt{n}\sqrt{n+1}\langle m|n\rangle + \sqrt{n+1}\sqrt{n+2}\langle m|n+2\rangle)$$

$$\boxed{\frac{\hbar}{2m\omega}(\sqrt{n}\sqrt{n-1}\delta_{m,n-2} + n\delta_{m,n} + \sqrt{n}\sqrt{n+1}\delta_{m,n} + \sqrt{n+1}\sqrt{n+2}\delta_{m,n+2})}$$

(v) $p^2 = \frac{m\hbar\omega}{2}(-a + a^\dagger)^2$

$p^2 = -\frac{m\hbar\omega}{2}(a \cdot a - a^\dagger \cdot a - a \cdot a^\dagger + a^\dagger \cdot a^\dagger)$

$p^2|n\rangle = -\frac{m\hbar\omega}{2}(a \cdot a|n\rangle - a^\dagger \cdot a|n\rangle - a \cdot a^\dagger|n\rangle + a^\dagger \cdot a^\dagger|n\rangle)$

$p^2|n\rangle = -\frac{m\hbar\omega}{2}(\sqrt{n}a|n-1\rangle - \sqrt{n}a^\dagger|n-1\rangle - \sqrt{n+1}a|n+1\rangle + \sqrt{n+1}a^\dagger|n+1\rangle)$

$p^2|n\rangle = -\frac{m\hbar\omega}{2}(\sqrt{n}\sqrt{n-1}|n-2\rangle - n|n\rangle - \sqrt{n}\sqrt{n+1}|n\rangle + \sqrt{n+1}\sqrt{n+2}|n+2\rangle)$

$\implies \langle m|p^2|n\rangle = -\frac{m\hbar\omega}{2}(\sqrt{n}\sqrt{n-1}\langle m|n-2\rangle - n\langle m|n\rangle - \sqrt{n}\sqrt{n+1}\langle m|n\rangle + \sqrt{n+1}\sqrt{n+2}\langle m|n+2\rangle)$

$$\boxed{\langle m|p^2|n\rangle = -\frac{m\hbar\omega}{2}(\sqrt{n}\sqrt{n-1}\delta_{m,n-2} - n\delta_{m,n} - \sqrt{n}\sqrt{n+1}\delta_{m,n} + \sqrt{n+1}\sqrt{n+2}\delta_{m,n+2})} \quad \blacksquare$$

References